Partial Differential Equations Examples Solutions

Introduction to Partial Differential Equations

Partial Differential Equations (PDEs) form the backbone of modern mathematical modeling in physics, engineering, finance, and biological sciences. Unlike ordinary differential equations, which involve functions of a single variable, PDEs describe how a function changes with respect to multiple independent variables. The objective of this article is to provide **partial differential equations examples solutions** that illustrate both the theory and the practical techniques used to solve them. By walking through real-world problems, we will demystify the process of solving PDEs, making the subject accessible to students, researchers, and professionals alike.

Whether you are studying heat transfer, wave propagation, or fluid dynamics, understanding how to approach a PDE and find its solution is crucial. We will explore fundamental methods such as separation of variables, Fourier series, and the method of characteristics, applying them to classic equations. Each example will be accompanied by a clear, step-by-step solution so that you can see exactly how the theory translates into practice.

What Are Partial Differential Equations?

A partial differential equation (PDE) is an equation that contains partial derivatives of an unknown function. The function typically depends on two or more independent variables (for example, space and time). PDEs are classified by their order (the highest derivative) and by their linearity. They appear in every branch of science and engineering.

For instance, the **heat equation** models temperature distribution over time, the **wave equation** describes vibrations, and **Laplace's equation** governs steady-state potentials. The solutions to these equations often involve boundary conditions and initial conditions, making each problem unique. Our goal is to present **partial**

differential equations examples solutions that demonstrate how to handle these conditions systematically.

Classification of Partial Differential Equations

Before diving into examples, it is helpful to understand the three main classes of second-order linear PDEs:

- **Elliptic** – e.g., Laplace's equation: $\nabla^2 u = 0$. Used for steady-state phenomena.
- **Parabolic** – e.g., the heat equation: $u_t = \alpha u_{xx}$. Used for diffusion-like processes.
- **Hyperbolic** – e.g., the wave equation: $u_{tt} = c^2 u_{xx}$. Used for wave propagation.

Each type requires a different solution strategy. The examples below will cover all three categories, providing concrete **partial differential equations examples solutions** that highlight these differences.

Method 1: Separation of Variables – The Heat Equation

The separation of variables technique assumes that the solution can be written as a product of functions, each depending on a single variable. Let us consider the one-dimensional heat equation on a rod of length L, with zero temperature at both ends:

PDE: $u_t = \alpha u_{xx}, \quad 0 < x < L, t > 0$

Boundary conditions: $u(0,t) = 0, u(L,t) = 0$

Initial condition: $u(x,0) = f(x)$

Step-by-Step Solution

1. **Assume a product solution:** $u(x,t) = X(x) T(t)$. Substitute into the PDE: $X T' = \alpha X'' T$.
2. **Separate variables:** Divide both sides by $\alpha X T$ to get $T' / (\alpha T) = X'' / X = -\lambda$, where λ is a constant.
3. **Solve the spatial ODE:** $X'' + \lambda X = 0$. With boundary conditions $X(0)=0, X(L)=0$, we get eigenvalues $\lambda_n = (n\pi / L)^2$ and eigenfunctions $X_n(x) = \sin(n\pi x / L)$.
4. **Solve the temporal ODE:** $T' + \alpha \lambda T = 0 \rightarrow T_n(t) = e^{-\alpha (n\pi/L)^2 t}$.
5. **General solution by superposition:** $u(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x / L) e^{-\alpha (n\pi/L)^2 t}$.
6. **Apply initial condition:** $u(x,0) = f(x) = \sum b_n \sin(n\pi x / L)$. The coefficients b_n are found using Fourier sine series: $b_n = (2/L) \int_0^L f(x) \sin(n\pi x / L) dx$.

This classic example of **partial differential equations examples solutions** demonstrates how an infinite series can represent the solution. For a specific f(x), you can compute the b_n and obtain an explicit series solution that can be approximated numerically.

Method 2: Fourier Series – The Wave Equation

The wave equation governs vibrations and is hyperbolic. Consider a string of length L fixed at both ends. The PDE is:

PDE: $u_{tt} = c^2 u_{xx}$, $0 < x < L$, $t > 0$

Boundary conditions: $u(0,t) = 0$, $u(L,t) = 0$

Initial conditions: $u(x,0) = f(x)$, $u_t(x,0) = g(x)$

Solution Using Separation of Variables

1. **Assume u(x,t) = X(x) T(t).** Substitution yields $X T'' = c^2 X'' T \to T'' / (c^2 T) = X'' / X = -\lambda$.
2. **Spatial problem:** $X'' + \lambda X = 0$, $X(0)=0$, $X(L)=0 \to$ eigenvalues $\lambda_n = (n\pi/L)^2$, $X_n = \sin(n\pi x / L)$.
3. **Temporal problem:** $T'' + c^2 \lambda T = 0 \to T_n(t) = A_n \cos(c n\pi t / L) + B_n \sin(c n\pi t / L)$.
4. **Superposition:** $u(x,t) = \sum_{n=1}^{\infty} [A_n \cos(c n\pi t / L) + B_n \sin(c n\pi t / L)] \sin(n\pi x / L)$.
5. **Apply initial position:** $u(x,0) = \sum A_n \sin(n\pi x/L) = f(x) \to A_n = (2/L) \int_0^L f(x) \sin(n\pi x/L)\, dx$.
6. **Apply initial velocity:** $u_t(x,0) = \sum (c n\pi / L) B_n \sin(n\pi x/L) = g(x) \to B_n = (2/(c n\pi)) \int_0^L g(x) \sin(n\pi x/L)\, dx$.

This is another fundamental example of **partial differential equations examples solutions** using eigenfunction expansion. For simple initial conditions such as a plucked string, the series converges quickly and provides an accurate representation of the wave motion.

Method 3: Solution of Laplace’s Equation – A Dirichlet Problem

Laplace's equation is elliptic and appears in steady-state heat conduction, electrostatics, and potential flow. Consider a rectangular plate $0 \le x \le a$, $0 \le y \le b$, with boundary conditions:

PDE: $u_{xx} + u_{yy} = 0$

Boundary conditions: $u(0,y)=0$, $u(a,y)=0$, $u(x,0)=0$, $u(x,b)=f(x)$

Solution Using Separation of Variables

1. **Assume u(x,y)= X(x) Y(y).** Substitution: $X'' Y + X Y'' = 0 \to X''/X = -Y''/Y = -\lambda$.
2. **Solve X problem:** $X'' + \lambda X = 0$, with $X(0)=0$, $X(a)=0 \to$ eigenvalues $\lambda_n = (n\pi/a)^2$, $X_n = \sin(n\pi x/a)$.
3. **Solve Y problem:** $Y'' - \lambda Y = 0 \to Y_n(y) = C_n \cosh(n\pi$

y / a) + D_n sinh(nπ y / a). Since Y(0)=0, we have C_n=0, so $Y_n = D_n$ sinh(nπ y / a).
4. **General solution:** $u(x,y) = \sum_{n=1}^{\infty} b_n \sin(n\pi x / a) \sinh(n\pi y / a)$.
5. **Apply top boundary:** $u(x,b) = \sum b_n \sin(n\pi x / a) \sinh(n\pi b / a) = f(x)$. Let $B_n = b_n \sinh(n\pi b/a)$. Then $B_n = (2/a) \int_0^a f(x) \sin(n\pi x / a)\, dx \rightarrow b_n = B_n / \sinh(n\pi b/a)$.

This example of **partial differential equations examples solutions** shows how hyperbolic functions naturally appear in bounded domains. The final solution is a series that satisfies all four boundary conditions.

Method 4: Method of

Characteristics – First-Order PDE

Not all PDEs are second-order. First-order linear PDEs are solved using the method of characteristics. Consider the transport equation:

PDE: $u_t + c\, u_x = 0$, with initial condition $u(x,0) = f(x)$.

Solution

1. **Parametrize the characteristic curves:** We look for curves (x(s), t(s)) along which u is constant. Set dx/ds = c, dt